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NEW PROOF OF A THEOREM OF OSGOOD'S IN THE CALCULUS OF VARIATIONS*

BY

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In the last number of the *Transactions*,† Professor OSGOOD has proved an important characteristic property of a strong minimum of an integral of the form

$$(1) \quad I = \int_{\tau_0}^{\tau_1} F(x, y, x', y') d\tau.$$

His proof, however, is rather complicated, and the following note is intended to give a simpler proof of the theorem.

§ 1.

Introduction of curvilinear coördinates.

Suppose the integral (1) is taken along a continuous curve C with continuously turning tangents:

$$C: \quad x = \phi(\tau), \quad y = \psi(\tau) \quad (\tau_0 \leq \tau \leq \tau_1)$$

joining two fixed points $A(\tau_0)$ and $B(\tau_1)$; further $\phi'^2 + \psi'^2 \neq 0$ in (τ_0, τ_1) .

Concerning the function $F(x, y, x', y')$ we make the same assumptions as OSGOOD on p. 277, l. c., except the assumption $F > 0$, which is not necessary for the present proof.

Now introduce instead of the rectangular coördinates x, y any curvilinear coördinates

$$(2) \quad u = U(x, y), \quad v = V(x, y),$$

where U, V are single-valued functions with continuous first and second derivatives in a region T of the x, y -plane containing the curve C ; in the same region their Jacobian is supposed $\neq 0$. Interpret u, v as the rectangular co-

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† *Transactions of the American Mathematical Society*, vol. 2, p. 273.

ordinates of a point in a u, v -plane and denote by T', C', A', B' the images of T, C, A, B respectively. Suppose further that the correspondence between T and T' is a one-to-one correspondence and that accordingly the inverse functions

$$(3) \quad x = X(u, v), \quad y = Y(u, v),$$

are single-valued functions with continuous first and second derivatives in T' and

$$D = \frac{\partial(X, Y)}{\partial(u, v)} \neq 0$$

in T' .

Then the integral I is changed into

$$(4) \quad I' = \int_{\tau_0}^{\tau_1} G(u, v, u', v') d\tau,$$

the function G of the four arguments u, v, u', v' being defined by

$$(5) \quad G(u, v, u', v') \equiv F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v'),$$

where $X_u = \partial X / \partial u$, etc.

The integral I' is taken, in the u, v -plane, along the image C' of C .

From $I = I'$ it follows that if the curve C minimize the integral I , its image C' will minimize I' , and vice versa; and if C be an extremal for I , C' must be an extremal for I' , and vice versa. Further WEIERSTRASS'S function F_1 is an invariant for the above transformation, viz., if we denote the corresponding function derived from G by G_1 , we obtain easily

$$(6) \quad G_1 = D^2 F_1.$$

Finally WEIERSTRASS'S E -function is an absolute invariant, i. e., if we denote the new E -function by E' we have:

$$(7) \quad \begin{cases} E'(u, v; u', v'; \bar{u}', \bar{v}') = E(x, y; x', y'; \bar{x}', \bar{y}') \\ \text{where} \\ x' = X_u u' + X_v v', \quad \bar{x}' = X_u \bar{u}' + X_v \bar{v}', \\ y' = Y_u u' + Y_v v', \quad \bar{y}' = Y_u \bar{u}' + Y_v \bar{v}', \end{cases}$$

as follows immediately from (5).

§ 2.

Proof of Osgood's theorem.

Now let

$$(8) \quad x = \phi(t, a), \quad y = \psi(t, a)$$

be a set of extremals for the integral I , satisfying the following conditions:

1. The functions ϕ, ψ are single-valued functions of t, a with continuous first and second derivatives in the region:

$$T_0 - \epsilon \leq t \leq T_1 + \epsilon, \quad |a - a_0| \leq \kappa \quad (\epsilon > 0).$$

2. The extremal $C_0: x = \phi(t, a_0), y = \psi(t, a_0)$ has no multiple point for $T - \epsilon \leq t \leq T_1 + \epsilon$, and passes through the two given points $A(t_0)$ and $B(t_1)$ where $T_0 < t_0 < t_1 < T_1$.

3. If we denote by $\Delta(t, a)$ the Jacobian

$$\Delta(t, a) = \frac{\partial(\phi, \psi)}{\partial(t, a)},$$

then

$$(9) \quad \Delta(t, a_0) \neq 0 \quad \text{in} \quad (T_0 - \epsilon, T_1 + \epsilon).$$

4. The inequality

$$(10) \quad F_1(\phi(t, a_0), \psi(t, a_0), \cos \lambda, \sin \lambda) > 0$$

holds for every t of the interval $T_0 - \epsilon \leq t \leq T_1 + \epsilon$ and for every λ .

Under these circumstances if we denote by R_κ the region:

$$R_\kappa: \quad T_0 \leq t \leq T_1, \quad |a - a_0| \leq \kappa,$$

and denote by S_κ the image of R_κ in the x, y -plane, then κ can be taken so small that

1. $\Delta(t, a) \neq 0$ in R_κ .

2. $F_1(x, y, \cos \lambda, \sin \lambda) > 0$ for every x, y in S_κ and for every λ .

3. The correspondence between R_κ and S_κ is a *one-to-one correspondence*.*

The inverse functions

$$t = t(x, y), \quad a = a(x, y)$$

will then be single-valued with continuous first and second derivatives in S_κ and their Jacobian will be $\neq 0$ in S_κ .

Then S_κ is a "field" surrounding the arc AB of the extremal C_0 , and if

$$C: \quad x = \bar{\phi}(\tau), \quad y = \bar{\psi}(\tau), \quad (\tau_0 \leq \tau \leq \tau_1),$$

be any other curve with continuously turning tangents drawn from A to B in S_κ , for which $\bar{\phi}'^2 + \bar{\psi}'^2 \neq 0$, then WEIERSTRASS'S theorem holds, according to which

$$(11) \quad \Delta I = I_{\bar{C}}(AB) - I_C(AB) = \int_{\tau_0}^{\tau_1} E(x, y; x', y'; \bar{x}', \bar{y}') d\tau,$$

* See KNESER, *Lehrbuch der Variationsrechnung*, § 14, and OSGOOD, l. c., p. 278. Both proofs have to be supplemented by the following preliminary lemma: If for every κ , however small, there existed points (x, y) in S_κ to which correspond in R_κ at least two distinct points (t', a') and (t'', a'') , then there must exist, in R_κ , a point (t_2, a_0) such that in every vicinity of it pairs of distinct points can be found whose images in the x, y -plane coincide.

where x', y' refer to the extremal of the set (8) passing through x, y , and $\bar{x}', \bar{y}'$ to the curve $\bar{C}$. Hence WEIERSTRASS infers that $\Delta I > 0$ by making use of the following theorem † connecting the functions E and F_1 :

$$(12) \quad E(x, y; \cos \vartheta, \sin \vartheta; \cos \bar{\vartheta}, \sin \bar{\vartheta}) = (1 - \cos \omega) F_1(x, y, \cos \vartheta^*, \sin \vartheta^*),$$

where $\omega = \bar{\vartheta} - \vartheta$; ϑ^* is an angle between ϑ and $\bar{\vartheta}$; and the angles are so measured that $|\omega| \equiv \pi$.

Now let $0 < h < \kappa$ and use S_h in the analogous signification as S_κ ; then OSGOOD'S theorem may be stated as follows: *There exists a positive quantity depending upon h, ϵ_h , such that for every curve C joining A and B drawn in the interior of S_κ but not wholly contained in S_h ,*

$$\Delta I > \epsilon_h.$$

To prove this theorem it is now only necessary to introduce instead of x, y the curvilinear coördinates

$$(13) \quad u = t(x, y), \quad v = a(x, y)$$

which satisfy for the regions S_κ and R_κ the conditions of § 1, and to make use of the remarks made there.

Accordingly the extremals for the integral I' are the lines $v = \text{const.}$, and, therefore, in the u, v -plane WEIERSTRASS'S theorem takes the form:

$$\begin{aligned} \Delta I = \Delta I' &= \int_{s_0}^{s_1} E'(u, v; \cos 0, \sin 0; \cos \omega, \sin \omega) ds \\ &= \int_{s_0}^{s_1} (1 - \cos \omega) G_1(u, v, \cos \vartheta^*, \sin \vartheta^*) ds. \end{aligned}$$

These integrals are taken along the image $\bar{C}'$ in the u, v -plane (i. e., t, a -plane) of the curve $\bar{C}$; ω denotes the angle between the positive tangent to $\bar{C}'$ in the point u, v and the positive u -axis; and s is the arc of the curve $\bar{C}'$.

But from (9), (10), and (6) it follows that we can assign a positive quantity m such that

$$G_1(u, v, \cos \lambda, \sin \lambda) \equiv m > 0$$

for every u, v in R_κ and for every λ . Therefore

$$\Delta I \equiv m \int_{s_0}^{s_1} (1 - \cos \omega) ds,$$

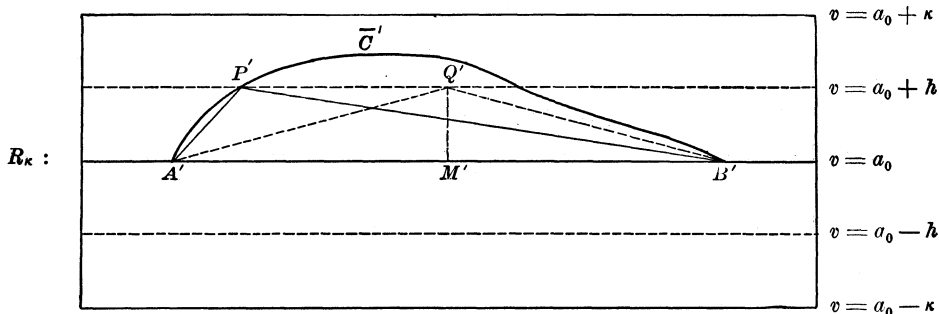
that is

$$\Delta I \equiv m (l - (t_1 - t_0)),$$

if l denotes the length of the curve $\bar{C}'$.

† *Lectures on the Calculus of Variations*, 1882.

Now suppose that $\bar{C}$ is not wholly contained in the interior of S_h and therefore passes through a point P of one of the two extremals $a = a_0 \pm h$, then $\bar{C}'$ passes through a point P' whose ordinate is $v = a_0 \pm h$. Hence l is greater



than or equal to the length of the broken line $A'P'B'$. But if we choose Q' on the same line $v = \text{const.}$ as P' so that $A'Q' = B'Q'$, then

$$A'P'B' \equiv A'Q'B',$$

and therefore

$$\Delta I \equiv 2m \left[\sqrt{h^2 + \left(\frac{t_1 - t_0}{2} \right)^2} - \frac{(t_1 - t_0)}{2} \right],$$

which proves OSGOOD's theorem.*

OSGOOD's theorem can easily be extended to the case where one endpoint, say B , is fixed, the other, A , movable on a given curve. For this purpose, it is only necessary to choose for the new coördinates KNESER's curvilinear coördinates u, v (KNESER, § 16); a slight modification of the above reasoning leads then to the inequality

$$\Delta I \equiv m [\sqrt{h^2 + (u_1 - u_0)^2} - (u_1 - u_0)].$$

* OSGOOD bases his proof upon the following lemma: "Let $f(x)$ be a single valued continuous function of x in the interval $a \leq x \leq b$, and let $f(x)$ have a continuous derivative $f'(x)$ at all points of this interval. Let $a < l \leq b$ and $|f(l) - f(a)| = L > 0$. Then

$$\int_a^l f'(x)^2 dx \geq \frac{L^3}{54(b-a)^2},$$

provided that $L < 3\sqrt{2}(b-a)$."

This lemma may be proved as follows: Since the value of the integral does not change if $f(x)$ is replaced by $M \pm f(x)$, M being a constant, we may confine ourselves to functions $f(x)$ for which $f(a) = 0, f(l) = L$. This remark reduces the question to the problem of minimizing the above integral with these given initial values. The solution is $y = L(x-a)/(l-a)$; it satisfies WEIERSTRASS' sufficient conditions for a minimum, and furnishes for the integral the minimum value: $(L^2)/(l-a) \equiv (L^2)/(b-a)$, which under the above inequality for L is greater than $\frac{1}{54}(L^3)/(b-a)^2$.

It is, however, to be remembered that the introduction of KNESER's coördinates presupposes that

$$F(\phi(t, a), \psi(t, a), \phi'(t, a), \psi'(t, a)) > 0$$

in the region R_κ .

UNIVERSITY OF CHICAGO,
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